Conservation Laws

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Lecture 2
Conservation laws

- Basic concepts that allow us to explain the change of the state of a system with time, if we know its initial condition, its boundary conditions, and its internal dynamics
- Mass conservation: mass of a closed system is conserved
- Momentum conservation: momentum is constant for an isolated system and otherwise can only be changed through the action of a force following Newton’s second law
- Energy: The total energy (potential+kinetic+internal) is constant for an isolated system and otherwise can only be changed by heat transfer or work exchange
Mass conservation

Conservation laws can be applied over a lumped discrete system (air in room), integrated over a finite system (integral forms), or applied at a “point”, i.e. over a very small volume (differential form of the equations)

mass in – mass out = change of mass of our control volume (per unit time)
For a closed system:  mass in = mass out=0

\[ \frac{dM}{dt} = Q_{in} - Q_{out} + S \]

Lumped

M: mass of CO\(_2\) in this room

\(Q_{in,out}\): mass flux rates of CO\(_2\) into and out from this room

S: sources and sinks of CO\(_2\) (humans breathing out, etc.)
Mass conservation

mass in – mass out = change of mass of our control volume (per unit time)

\[ \vec{U} = u \vec{i} + v \vec{j} + w \vec{k} \]

Mass conservation applied to a constituent of concentration \( C \) (M/L\(^3\))
... for an advected constituent with volumetric concentration \( C = \frac{M}{(dx \, dy \, dz)} = \frac{M}{dV} \)

**Fluxes from each face**

\[
\left( C + \frac{\partial C}{\partial z} \right) \left( w + \frac{\partial w}{\partial z} \right) \, dx \, dy \\
\left( C + \frac{\partial C}{\partial y} \right) \left( v + \frac{\partial v}{\partial y} \right) \, dx \, dz \\
\left( C + \frac{\partial C}{\partial x} \right) \left( u + \frac{\partial u}{\partial x} \right) \, dy \, dz \\
\left( C - \frac{\partial C}{\partial y} \right) \left( v - \frac{\partial v}{\partial y} \right) \, dx \, dz \\
\left( C - \frac{\partial C}{\partial z} \right) \left( w - \frac{\partial w}{\partial z} \right) \, dx \, dy
\]

**Taylor series expansion, origin is center of mass / red point**

\[
\frac{\partial C}{\partial t} + \nabla \cdot (C \vec{U}) = 0
\]
If C is the density (concentration of all constituents)

→ Continuity equation

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{U}) = 0 \]

What is it for an incompressible flow?
When C can also be diffused (Fick’s law)

\[
F_{x-\frac{dx}{2}} = -\kappa \left( \frac{\partial C}{\partial x} \right)_{x-\frac{dx}{2}} dydz
\]

Taylor series expansion

\[
-\kappa \left( \frac{\partial C}{\partial x} - \frac{\partial^2 C}{\partial x^2} \frac{dx}{2} \right) dydz
\]

To add to Eq. with advection

\[
\frac{\partial C}{\partial t} + \nabla \cdot (CU) = \kappa \nabla^2 C
\]

Advection-Diffusion Equation
When C is the density of water vapor $\rho_v$

$$\frac{\partial \rho_v}{\partial t} + \nabla \cdot (\rho_v \vec{U}) = \kappa \nabla^2 \rho_v + \text{Sources} / \text{Sinks}$$

Replacing $\rho_v = q \rho_{moist\ air}$ to get the equation of conservation of specific humidity $q = \text{mass water vapor} / \text{mass moist air}$ and using the continuity equation yields

$$\frac{\partial q}{\partial t} + \vec{U} \cdot \nabla q = \kappa \nabla^2 q + S(\text{evaporation}, \text{Condensation})$$

Writing mass conservation equations in terms of mass ratios is better than in terms of volumetric concentrations since mass ratios are conserved for any closed system regardless of its compressibility since no gradients of fluid density appear in the equation (material derivative=0 for a closed system).

Volumetric concentrations on the other hand are not conserved for a closed system undergoing a change of density.

Note: incompressible flow $\neq$ density is constant
Momentum conservation

- Newton’s second law: \( \mathbf{F} = m \mathbf{a} = m \frac{d\mathbf{U}}{dt} \) (bold or arrow above \( \to \) vector)
  - Applied to a point mass
- We can also apply it to a closed fluid particle (cst mass) at its center of mass
- *This fluid particle will most likely be moving \( \to \) Lagrangian frame of reference*
- We cannot apply it as is to a fixed fluid element: mass is changing: we have to account for flux of momentum in and out by the flow
- *Fixed fluid element with flow in and out \( \to \) Eulerian frame of reference*
Momentum conservation: Forces

- **Real Forces**
  - Pressure (Gradient)
  - Viscous Forces
  - Gravity
  - *Inertial forces: arise from spatial variability of momentum in the fluid: \( \text{mom}_{in} - \text{mom}_{out} \)*
  - Sufficient in an inertial, non-rotating frame of reference

- **Apparent Forces: due to RCS**
  - Coriolis Force
  - Centrifugal Force
Momentum conservation without apparent forces → 3 equations (vector equation)

\[
\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}
\]

\[
\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 v}{\partial y^2} + \mu \frac{\partial^2 w}{\partial z^2}
\]

\[
\rho \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = -\frac{\partial p}{\partial y} + \mu \frac{\partial^2 v}{\partial x^2} + \mu \frac{\partial^2 v}{\partial y^2} + \mu \frac{\partial^2 v}{\partial z^2}
\]

\[
\rho \left( \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial z^2} - \rho g^*
\]

\(\mu = \) dynamic viscosity

Stokes assumption for a Newtonian fluid
Momentum conservation without apparent forces

\[ \frac{\partial \vec{U}}{\partial t} + U \cdot \nabla \vec{U} = -\frac{1}{\rho} \nabla p + \frac{1}{\rho} \nabla \cdot \vec{\tau} + \vec{g} \]

If we were following a fluid particle

Material derivative

\[ \frac{D\vec{U}}{Dt} = \frac{\partial \vec{U}}{\partial t} + U \cdot \nabla \vec{U} = -\frac{1}{\rho} \nabla p + \frac{1}{\rho} \nabla \cdot \vec{\tau} + \vec{g} \]

e.g.

\[ \frac{Dw}{Dt} = \frac{\partial w}{\partial t} + u \frac{dw}{dx} + v \frac{dw}{dy} + w \frac{dw}{dz} \]

\[ p = \text{pressure} \]

\[ \vec{\tau} = \text{viscous stress tensor (2^{nd} order)} = \mu \nabla \vec{U} \text{ (model)} \]

\[ \vec{g} = \text{gravity} = -g \cdot \vec{k} \]

\[ \frac{\partial \vec{U}}{\partial t} + U \cdot \nabla \vec{U} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{U} + \vec{g} \]

\( \nu = \text{kinematic viscosity} \)
Pseudo-Forces

- Geocentric reference frame: frame at rest with respect to the rotating earth
- The geocentric frame is non-inertial (it has an acceleration due to its rotation) to apply Newton’s laws, the acceleration of the coordinate system should be accounted for via fictitious forces or pseudo-forces or apparent forces that we introduce in Newton’s second law
- What we are doing is trying to translate the change in absolute motion of the fluid to a change in its relative motion with respect to geocentric frame
The Centrifugal Force & Apparent Gravity

\( \vec{g} \) : real gravity, points towards the center of the earth

Centrifugal force: \(- \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) = \Omega^2 \vec{R}_A\) points away from the rotation axis

\( \Omega = |\vec{\Omega}| = 2\pi \text{ rad day}^{-1} = 7.292 \times 10^{-5} \text{ s}^{-1} = \text{speed of earth's rotation} \)

\[ |\vec{g}^*| > \Omega^2 |\vec{R}_A|, \quad \text{or we would fly off the earth} \]

Therefore, \( \vec{g}^* + \Omega^2 \vec{R}_A \) points towards the surface, with a direction that will be \( \perp \) to the surface!

Why? (hint, think about the sphericity of the earth)

\[ \vec{g}^* + \Omega^2 \vec{R}_A = \vec{g} = -g \vec{k} = -9.81 \vec{k} \]

is called the apparent gravity and used instead of the separate 2 forces

\( \vec{k} \) is the unit vector pointing upward in the direction perpendicular to the surface
The Coriolis Force:

\[-2\Omega \times \vec{U}\]

Fig. 1. Since the Coriolis force is the cross product between the rotational vector of the earth (\(\Omega\)) and the velocity vector, it will take its maximal values for motions perpendicular to the earth’s axis (a) and vanish for motions parallel to the earth’s axis (b). Affected by a maximal Coriolis force is, for example, air rising equatorward (or sinking poleward) in the midlatitudes. The Coriolis force vanishes for air rising poleward (or sinking equatorward). Horizontal west–east winds on the equator are indeed affected by the Coriolis force, although the deflection is completely in the vertical direction.
Momentum conservation with apparent forces

\[ \frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U} = -\frac{1}{\rho} \nabla p + \mathbf{v} \nabla^2 \mathbf{U} + \mathbf{g} - 2\Omega \times \mathbf{u} - \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) \]

[i,j,k] refer to [eastward, northward, vertical]

Coriolis Force = \(-2\Omega \times \mathbf{U} \approx -f_c \mathbf{k} \times \mathbf{U}\)

where the Coriolis parameter is \(f_c = 2\Omega \sin(\phi)\)

\(\Omega = 7.292 \times 10^{-5} \text{ s}^{-1}\) = speed of earth's rotation

\(\phi\) is the latitude
Hydrostatic equilibrium

\[
\frac{dp}{dz} = -\rho g \\
p - p_0 = -\rho g(z - z_0)
\]
Other forces

Ideal Gas: \( p = \rho RT \Rightarrow \) Heat exchanges \( \Rightarrow \) density changes \( \Rightarrow \) buoyant forces

Boussinesq approximation = these changes cause small changes in \( \rho \) and \( p \) from their average values \( \bar{\rho} \) and \( \bar{p} \)

\[ \rho' \ll \bar{\rho} \quad \text{where} \quad \rho = \bar{\rho} + \rho' \quad \text{AND} \quad p' \ll \bar{p} \quad \text{where} \quad p = \bar{p} + p' \]

such that

\[
-\frac{\partial \rho}{\partial z} - \rho g = -\frac{\partial \bar{\rho}}{\partial z} - \frac{\partial p'}{\partial z} - \bar{\rho} g - \rho' g \\
\text{from} \quad \rho \left( \frac{\partial w}{\partial t} + \bar{U} \cdot \nabla w \right) = -\nabla p + \mu \nabla^2 w - \rho g
\]

Usually we can ignore \( \rho' \approx 0 \) in all terms, but when atmosphere close to hydrostatic equilibrium \( -\frac{\partial \bar{\rho}}{\partial z} \approx \bar{\rho} g, -\rho' g \) and \( \frac{\partial p'}{\partial z} \) become important in vertical force balance, but \( \rho' \) is not important in inertial term.

Dividing by \( \bar{\rho} \approx \rho \) and replacing in the z-momentum equation \( \Rightarrow \)

\[
\frac{\partial w}{\partial t} + \bar{U} \cdot \nabla w = -\frac{1}{\bar{\rho}} \frac{\partial p}{\partial z} \left( 1 + \frac{\rho'}{\bar{\rho}} \right) g + \nu \nabla^2 w
\]

Boussinesq approximation \( \Rightarrow \bar{\rho} \approx \rho \), except when multiplied by \( g \), keep \( \rho' \)
Back to Urban

It turns out that this new term: \(- \frac{\rho'}{\bar{\rho}} g\)

has major significance for urban microclimates, and many environmental flows. Recall, small deviations \(\rho'\) from \(\bar{\rho}\), are often due to heating/cooling of air, mainly at the surface. Let's now study the heat conservation equation to understand this.

How do the heat and momentum equations couple?
How and where are the heating/cooling bouyant forces generated?
What do they imply for the urban environment?
... in the next few lectures
Sensible Heat Conservation Equation

Air flow near earth surface and liquid water flow \(\sim\) incompressible work \(\rightarrow\) changes kinetic and potential energies (if no vertical motion)

heat exchanges \(\rightarrow\) change internal energy \(\Rightarrow\) energy conservation \(\approx\) heat conservation

\[
dq = dh = c_p \, dT
\]

Discrete system:

\[
m \frac{dq}{dt} = mc_p \frac{dT}{dt} = \rho \, dV \, c_p \frac{dT}{dt}
\]

infinitesimal point

\[
\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} = \frac{k}{\rho c_p} \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right)
\]

Heat diffusion, similar to mass diffusion, more in lecture 3

\[
\frac{dT}{dt} + u \cdot \nabla T = \frac{k}{\rho c_p} \nabla^2 T + \text{Radiation} + \text{Latent Heat Conversion}
\]

Ok when buoyant forces weak
Sensible Heat Conservation Equation

Full Equation: \[ \frac{\partial T}{\partial t} + u \cdot \nabla T = \frac{k}{\rho c_p} \nabla^2 T - \frac{1}{\rho c_p} (\nabla \cdot \vec{R}) - \frac{L_e}{\rho c_p} E \]

\( k \) = thermal conductivity
\( \rho \) = density
\( c_p \) = specific heat of air
\( \frac{k}{\rho c_p} = \alpha \) = the thermal diffusivity

where \( \vec{R} \) is the net radiation vector
\( L_e \) is the latent heat of vaporization
and \( E \) is the evaporation rate (negative if condensation)

The last term is the exchange of sensible and latent heat
Recommended readings

- Roland Stull: An Introduction to Boundary Layer Meteorology
- Alexander Smits: A physical introduction to fluid mechanics
- Benoit Cushman-Roisin: Environmental Fluid Mechanics
  [http://engineering.dartmouth.edu/~cushman/courses/engs151.html](http://engineering.dartmouth.edu/~cushman/courses/engs151.html)